

Symbolic stochastic dynamical systems viewed as binary N -step Markov chains

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A theory of systems with long-range correlations based on the consideration of *binary N -step Markov chains* is developed. In the model, the conditional probability that the i th symbol in the chain equals zero (or unity) is a linear function of the number of unities among the preceding N symbols. The correlation and distribution functions as well as the variance of the number of symbols in the words of arbitrary length L are obtained analytically and numerically. A self-similarity of the studied stochastic process is revealed and the similarity group transformation of the chain parameters is presented. The diffusion Fokker-Planck equation governing the distribution function of the L words is explored. If the persistent correlations are not extremely strong, the distribution function is shown to be the Gaussian with the variance being nonlinearly dependent on L . The applicability of the developed theory to the coarse-grained written and DNA texts is discussed.

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I. INTRODUCTION

The problem of systems with long-range spatial and/or temporal correlations (LRCS) is one of the topics of intensive research in modern physics, as well as in the theory of dynamical systems and the theory of probability. The LRC systems are usually characterized by a complex structure and contain a number of hierarchic objects as their subsystems. The LRC systems are the subject of study in physics, biology, economics, linguistics, sociology, geography, psychology, etc. [1–4]. At the present time, there is no generally accepted theoretical model that adequately describes the dynamical and statistical properties of the LRC systems. Attempts to describe the behavior of the LRCS in the framework of the Tsallis nonextensive thermodynamics [5,6] were undertaken in Ref. [7]. However, the nonextensive thermodynamics is not well-grounded and requires the construction of the additional models which could clarify the properties of the LRC systems.

One of the efficient methods to investigate the correlated systems is based on a decomposition of the space of states into a finite number of parts labeled by definite symbols. This procedure referred to as coarse graining is accompanied by the loss of short-range memory between states of system but does not affect and does not damage its robust invariant statistical properties on large scales. The most frequently used method of the decomposition is based on the introduction of two parts of the phase space. In other words, it consists in mapping the two parts of states onto two symbols, say 0 and 1. Thus the problem is reduced to investigating the statistical properties of the symbolic binary sequences. This method is applicable for the examination of both discrete and continuous systems.

One of the ways to get a correct insight into the nature of correlations consists in an ability of constructing a math-

ematical object (for example, a correlated sequence of symbols) possessing the same statistical properties as the initial system. There are many algorithms to generate long-range correlated sequences: the inverse Fourier transform [8], the expansion-modification Li method [9], the Voss procedure of consequent random addition [10], the correlated Levy walks [11], etc. [8]. We believe that, among the above-mentioned methods, using the Markov chains is one of the most important. We would like to demonstrate this statement in the present paper.

In the following sections, the statistical properties of the *binary many-step Markov chain* is examined. In spite of the long-time history of studying Markov sequences (see, for example [4,12,13] and references therein), the concrete expressions for the variance of sums of random variables in such strings have not yet been obtained. Our model operates with two parameters governing the conditional probability of the discrete Markov process, specifically with the memory length N and the correlation parameter μ . The correlation and distribution functions as well as the variance D being nonlinearly dependent on the length L of a word are derived analytically and calculated numerically. The nonlinearity of the $D(L)$ function reflects the existence of strong correlations in the system. The evolved theory is applied to the coarse-grained written texts and dictionaries, and to DNA strings as well.

Some preliminary results of this study were published in Ref. [14].

II. FORMULATION OF THE PROBLEM**A. Markov processes**

Let us consider a homogeneous binary sequence of symbols, $a_i = \{0,1\}$. To determine the *N -step Markov chain* we have to introduce the conditional probability $P(a_i | a_{i-N}, a_{i-N+1}, \dots, a_{i-1})$ of occurring the definite symbol a_i (for example, $a_i = 0$) after symbols $a_{i-N}, a_{i-N+1}, \dots, a_{i-1}$. Thus, it is necessary to define 2^N

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values of the P -function corresponding to each possible configuration of the symbols $a_{i-N}, a_{i-N+1}, \dots, a_{i-1}$. We suppose that the P function has the form

$$P(a_i=0|a_{i-N}, a_{i-N+1}, \dots, a_{i-1}) = \frac{1}{N} \sum_{k=1}^N f(a_{i-k}, k). \quad (1)$$

Here the value $f(a_{i-k}, k)/N$ is the contribution of the symbol a_{i-k} to the conditional probability of occurring the symbol zero at the i th site. Relation (1) corresponds to the additive influence of the previous symbols on the generated one. The homogeneity of the Markov chain is provided by the independence of the conditional probability Eq. (1) of the index i .

It is reasonable to assume the function f to be decreasing with an increase of the distance k between the symbols a_{i-k} and a_i in the Markov chain. However, for the sake of simplicity we consider here a steplike memory function $f(a_{i-k}, k)$ independent of the second argument k . As a result, the model is characterized by three parameters only, specifically by $f(0)$, $f(1)$, and N :

$$P(a_i=0|a_{i-N}, a_{i-N+1}, \dots, a_{i-1}) = \frac{1}{N} \sum_{k=1}^N f(a_{i-k}). \quad (2)$$

Note that the probability P in Eq. (2) depends on the numbers of symbols 0 and 1 in the N word but is independent of the arrangement of the elements a_{i-k} . We also suppose that

$$f(0) + f(1) = 1. \quad (3)$$

This relation provides the statistical equality of the numbers of symbols zero and unity in the Markov chain under consideration. In other words, the chain is nonbiased. Indeed, taking into account Eqs. (2) and (3) and the sequence of equations,

$$\begin{aligned} P(a_i=1|a_{i-N}, \dots, a_{i-1}) &= 1 - P(a_i=0|a_{i-N}, \dots, a_{i-1}) \\ &= \frac{1}{N} \sum_{k=1}^N f(\tilde{a}_{i-k}) \\ &= P(a_i=0|\tilde{a}_{i-N}, \dots, \tilde{a}_{i-1}), \end{aligned} \quad (4)$$

one can see the symmetry with respect to interchange $\tilde{a}_i \leftrightarrow a_i$ in the Markov chain. Here \tilde{a}_i is the symbol opposite to a_i , $\tilde{a}_i = 1 - a_i$. Therefore, the probabilities of occurring the words (a_1, \dots, a_L) and $(\tilde{a}_1, \dots, \tilde{a}_L)$ are equal to each other for any word length L . At $L=1$ this yields equal average probabilities that symbols 0 and 1 occur in the chain.

Taking into account the symmetry of the conditional probability P with respect to a permutation of symbols a_i [see Eq. (2)], we can simplify the notations and introduce the conditional probability p_k of occurring the symbol zero after the N word containing k unities, e.g., after the word $(\underbrace{11 \dots 1}_k \underbrace{00 \dots 0}_{N-k})$,

$$p_k = P \left[a_{N+1}=0 | \underbrace{11 \dots 1}_k \underbrace{00 \dots 0}_{N-k} \right] = \frac{1}{2} + \mu \left(1 - \frac{2k}{N} \right), \quad (5)$$

with the correlation parameter μ being defined by the relation

$$\mu = f(0) - \frac{1}{2}. \quad (6)$$

We focus our attention on the region of μ determined by the persistence inequality $0 < \mu < 1/2$. In this case, each of the symbols unity in the preceding N word promotes the birth of new symbol unity. Nevertheless, the major part of our results is valid for the antipersistent region $-1/2 < \mu < 0$ as well.

A similar rule for the production of an N word (a_1, \dots, a_N) that follows after a word $(a_0, a_1, \dots, a_{N-1})$ was suggested in Ref. [4]. However, the conditional probability p_k of the symbols a_N occurring does not depend on the previous ones in the model [4].

B. Statistical characteristics of the chain

In order to investigate the statistical properties of the Markov chain, we consider the distribution $W_L(k)$ of the words of definite length L by the number k of unities in them,

$$k_i(L) = \sum_{l=1}^L a_{i+l}, \quad (7)$$

and the variance of k ,

$$D(L) = \overline{k^2} - \bar{k}^2, \quad (8)$$

where

$$\overline{f(k)} = \sum_{k=0}^L f(k) W_L(k). \quad (9)$$

If $\mu=0$, one arrives at the known result for the noncorrelated Brownian diffusion,

$$D(L) = L/4. \quad (10)$$

We will show that the distribution function $W_L(k)$ for the sequence determined by Eq. (5) (with nonzero but not extremely close to $1/2$ parameter μ) is the Gaussian with the variance $D(L)$ nonlinearly dependent on L . However, at $\mu \rightarrow 1/2$ the distribution function can differ from the Gaussian.

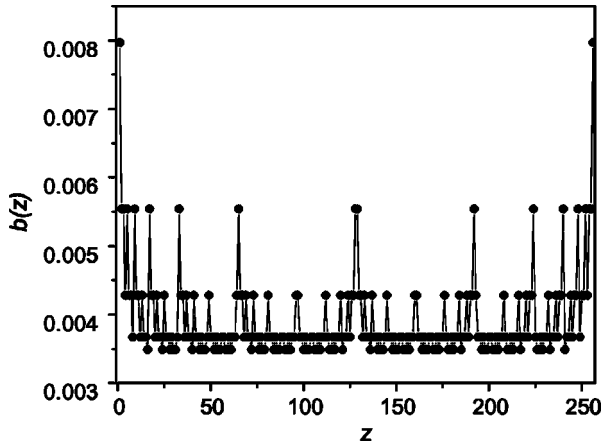


FIG. 1. The probability b of a word occurring (a_1, a_2, \dots, a_N) vs its number z expressed in the binary code, $z = \sum_{i=1}^N a_i \times 2^{i-1}$, for $N=8$, $\mu=0.4$.

C. Main equation

For the stationary Markov chain, the probability $b(a_1 a_2 \dots a_N)$ of occurring a certain word (a_1, a_2, \dots, a_N) satisfies the condition of compatibility for the Chapman-Kolmogorov equation (see, for example, Ref. [15]):

$$b(a_1 \dots a_N) = \sum_{a=0,1} b(a a_1 \dots a_{N-1}) \times P(a_N | a, a_1, \dots, a_{N-1}). \tag{11}$$

Thus, we have 2^N homogeneous algebraic equations for the 2^N probabilities b of the N words occurring and the normalization equation $\sum b = 1$. In the case under consideration, the set of equations can be substantially simplified owing to the following statement.

Proposition 1. The probability $b(a_1 a_2 \dots a_N)$ depends on the number k of unities in the N word only, i.e., it is independent of the arrangement of symbols in the word (a_1, a_2, \dots, a_N) .

This statement illustrated by Fig. 1 is valid owing to the chosen simple model (2), (5) of the Markov chain. It can be easily verified directly by substituting the obtained below solution (15) into the set (11). Note that, according to the Markov theorem, Eqs. (11) do not have other solutions [16].

Proposition 1 leads to the very important property of isotropy: any word (a_1, a_2, \dots, a_L) appears with the same probability as the inverted one, $(a_L, a_{L-1}, \dots, a_1)$.

Let us apply the set of Eqs. (11) to the word $(\underbrace{11\dots 1}_k \underbrace{00\dots 0}_{N-k})$:

$$b(\underbrace{11\dots 1}_k \underbrace{00\dots 0}_{N-k}) = b(\underbrace{011\dots 1}_k \underbrace{00\dots 0}_{N-k-1}) p_k + b(\underbrace{111\dots 1}_k \underbrace{00\dots 0}_{N-k-1}) p_{k+1}. \tag{12}$$

This yields the recursion relation for $b(k) = b(\underbrace{11\dots 1}_k \underbrace{00\dots 0}_{N-k})$,

$$b(k) = \frac{1-p_{k-1}}{p_k} b(k-1) = \frac{N-2\mu(N-2k+2)}{N+2\mu(N-2k)} b(k-1). \tag{13}$$

The probabilities $b(k)$ for $\mu > 0$ satisfy the sequence of inequalities,

$$b(0) = b(N) > b(1) = b(N-1) > \dots > b(N/2), \tag{14}$$

which is the reflection of persistent properties for the chain. At $\mu = 0$ all probabilities are equal to each other.

The solution of Eq. (11) is

$$b(k) = A \cdot \Gamma(n+k) \Gamma(n+N-k) \tag{15}$$

with the parameter n defined by

$$n = \frac{N(1-2\mu)}{4\mu}. \tag{16}$$

The constant A will be found below by normalizing the distribution function. Its value is

$$A = \frac{4^n}{2\sqrt{\pi}} \frac{\Gamma(1/2+n)}{\Gamma(n)\Gamma(2n+N)}. \tag{17}$$

III. DISTRIBUTION FUNCTION OF L WORDS

In this section we investigate the statistical properties of the Markov chain, specifically, the distribution of the words of definite length L by the number k of unities. The length L can also be interpreted as the number of jumps of some particle over an integer-valued 1D lattice or as the time of the diffusion imposed by the Markov chain under consideration. The form of the distribution function $W_L(k)$ depends, to a large extent, on the relation between the word length L and the memory length N . Therefore, the first thing we will do is to examine the simplest case $L=N$.

A. Statistics of N words

The value $b(k)$ is the probability that an N word contains k unities with a definite order of symbols a_i . Therefore, the probability $W_N(k)$ that an N word contains k unities with arbitrary order of symbols a_i is $b(k)$ multiplied by the number $C_N^k = N!/k!(N-k)!$ of different permutations of k unities in the N word,

$$W_N(k) = C_N^k b(k). \tag{18}$$

Combining Eqs. (15) and (18), we find the distribution function,

$$W_N(k) = W_N(0) C_N^k \frac{\Gamma(n+k)\Gamma(n+N-k)}{\Gamma(n)\Gamma(n+N)}. \tag{19}$$

The normalization constant $W_N(0)$ can be obtained from the equality $\sum_{k=0}^N W_N(k) = 1$,

$$W_N(0) = \frac{4^n}{2\sqrt{\pi}} \frac{\Gamma(n+N)\Gamma(1/2+n)}{\Gamma(2n+N)}. \quad (20)$$

Comparing Eqs. (15), (18)–(20), one can get Eq. (17) for the constant A in Eq. (15).

Note that the distribution $W_N(k)$ is an even function of the variable $\kappa = k - N/2$,

$$W_N(N-k) = W_N(k). \quad (21)$$

This fact is a direct consequence of the above-mentioned statistical equivalence of zeros and unities in the Markov chain being considered. Let us analyze the distribution function $W_N(k)$ for different relations between the parameters N and μ .

1. Limiting case of weak persistence, $n \gg 1$

In terms of the correlation parameter μ , this limiting case corresponds to the values of μ not very close to $1/2$,

$$\frac{1-2\mu}{4\mu} \gg \frac{1}{N}. \quad (22)$$

This inequality can be rewritten via the f function [see Eqs. (2)–(6)],

$$\frac{f(1)}{f(0)-f(1)} \gg \frac{1}{N}. \quad (23)$$

In the absence of correlations, $n \rightarrow \infty$, Eq. (19) and the Stirling formula yield the Gaussian distribution at $k, N, N - k \gg 1$. Given the persistence is not too strong,

$$n \gg 1, \quad (24)$$

one can also obtain the Gaussian form for the distribution function,

$$W_N(k) = \frac{1}{\sqrt{2\pi D(N)}} \exp\left\{-\frac{(k-N/2)^2}{2D(N)}\right\}, \quad (25)$$

with the μ -dependent variance,

$$D(N) = \frac{N(N+2n)}{8n} = \frac{N}{4(1-2\mu)}. \quad (26)$$

Equation (25) says that the N words with equal numbers of zeros and unities, $k = N/2$, are most probable. Note that the persistence results in an increase of the variance $D(N)$ with respect to its value $N/4$ at $\mu = 0$. In other words, the persistence is conducive to the intensification of the diffusion. Inequality $n \gg 1$ gives $D(N) \ll N^2$. Therefore, despite the increase of $D(N)$, the fluctuations of $(k - N/2)$ of the order of N are exponentially small.

2. Intermediate case, $n \geq 1$

If the parameter n is an integer of the order of unity, the distribution function $W_N(k)$ is a polynomial of degree $2(n - 1)$. In particular, at $n = 1$, the function $W_N(k)$ is constant,

$$W_N(k) = \frac{1}{N+1}. \quad (27)$$

At $n \neq 1$, $W_N(k)$ has a maximum in the middle of the interval $[0, N]$.

3. Limiting case of strong persistence

If the parameter n satisfies the inequality,

$$n \ll \ln^{-1}N, \quad (28)$$

or

$$1 - 2\mu \ll 1/N \ln(N), \quad f(1) \ll 1/N \ln(N), \quad (29)$$

then one can neglect the parameter n in the arguments of the functions $\Gamma(n+k)$, $\Gamma(n+N)$, and $\Gamma(n+N-k)$ in Eq. (19). In this case, the distribution function $W_N(k)$ assumes its maximal values at $k = 0$ and $k = N$,

$$W_N(1) = W_N(0) \frac{nN}{N-1} \ll W_N(0). \quad (30)$$

Formula (30) describes the sharply decreasing $W_N(k)$ as k varies from 0 to 1 (and from N to $N-1$). Then, at $1 < k < N/2$, the function $W_N(k)$ decreases more slowly with an increase in k ,

$$W_N(k) = W_N(0) \frac{nN}{k(N-k)}. \quad (31)$$

At $k = N/2$, the probability $W_N(k)$ achieves its minimal value,

$$W_N\left(\frac{N}{2}\right) = W_N(0) \frac{4n}{N}. \quad (32)$$

It follows from normalization (20) that the values $W_N(0) = W_N(N)$ are approximately equal to $1/2$. Neglecting the terms of the order of n^2 , one gets

$$W_N(0) = \frac{1}{2}(1 - n \ln N). \quad (33)$$

In the straightforward calculation using Eqs. (8) and (31) the variance D is

$$D(N) = \frac{N^2}{4} - \frac{nN(N-1)}{2}. \quad (34)$$

Thus, the variance $D(N)$ is equal to $N^2/2$ in the leading approximation in the parameter n . This fact has a simple explanation. The probability of occurrence of the N word containing N unities is approximately equal to $1/2$. So, the relations $\bar{k}^2 \approx N^2/2$ and $\bar{k}^2 = N^2/4$ give Eq. (34). The case of strong persistence corresponds to the so-called ballistic regime of diffusion: if we chose randomly some symbol a_i in the sequence, it will be surrounded by the same symbols with the probability close to unity.

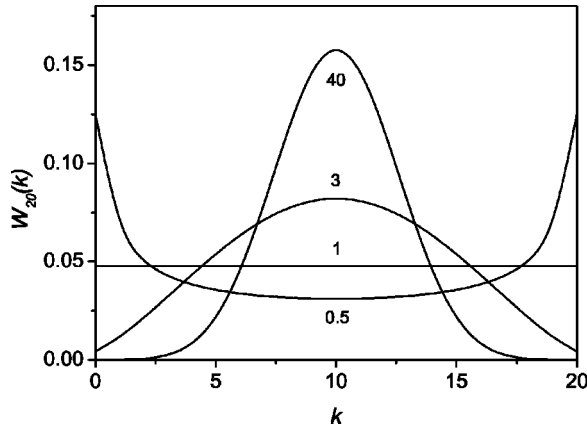


FIG. 2. The distribution function $W_N(k)$ for $N=20$ and different values of the parameter n shown near the curves.

The evolution of the distribution function $W_N(k)$ from the Gaussian form to the inverse one with a decrease of the parameter n is shown in Fig. 2. In the interval $\ln^{-1}N < n < 1$ the curve $W_N(k)$ is concave and the maximum of function $W_N(k)$ inverts into minimum. At $N \gg 1$ and $\ln^{-1}N < n < 1$, the curve remains a smooth function of its argument k as shown by curve with $n=0.5$ in Fig. 2. Below, we will not consider this relatively narrow region of the change in the parameter n .

Formulas (25), (26), (31), (33) and (34) describe the statistical properties of L words for the fixed “diffusion time” $L=N$. It is necessary to examine the distribution function $W_L(k)$ for the general situation, $L \neq N$. We start the analysis with $L < N$.

B. Statistics of L words with $L < N$

1. Distribution function $W_L(k)$

The distribution function $W_L(k)$ at $L < N$ can be given as

$$W_L(k) = \sum_{i=k}^{k+N-L} b(i) C_L^k C_{N-L}^{i-k}. \quad (35)$$

This equation follows from the consideration of N words consisting of two parts,

$$\left(\underbrace{a_1, \dots, a_{N-L}}_{i-k \text{ unities}}, \underbrace{a_{N-L+1}, \dots, a_N}_{k \text{ unities}} \right). \quad (36)$$

The total number of unities in this word is i . The right-hand part of the word (L subword) contains k unities. The remaining $(i-k)$ unities are situated within the left-hand part of the word [within $(N-L)$ subword]. The multiplier $C_L^k C_{N-L}^{i-k}$ in Eq. (35) takes into account all possible permutations of the symbols “1” within the N word on condition that the L subword always contains k unities. Then we perform the summation over all possible values of the number i . Note that Eq. (35) is a direct consequence of the proposition 1 formulated in Sec. II C.

The straightforward summation in Eq. (35) yields the following formula that is valid at any value of the parameter n :

$$W_L(k) = W_L(0) C_L^k \frac{\Gamma(n+k)\Gamma(n+L-k)}{\Gamma(n)\Gamma(n+L)}, \quad (37)$$

where

$$W_L(0) = \frac{4^n}{2\sqrt{\pi}} \frac{\Gamma(1/2+n)\Gamma(n+L)}{\Gamma(2n+L)}. \quad (38)$$

It is of interest to note that the parameter of persistence μ and the memory length N are presented in Eqs. (37) and (38) via the parameter n only. This means that the statistical properties of the L words with $L < N$ are defined by this single “combined” parameter.

In the limiting case of weak persistence, $n \gg 1$, at $k, L-k \gg 1$, Eq. (37) along with the Stirling formula give the Gaussian distribution function,

$$W_L(k) = \frac{1}{\sqrt{2\pi D(L)}} \exp\left\{-\frac{(k-L/2)^2}{2D(L)}\right\} \quad (39)$$

with the variance $D(L)$,

$$D(L) = \frac{L}{4} \left(1 + \frac{L}{2n}\right) = \frac{L}{4} \left[1 + \frac{2\mu L}{N(1-2\mu)}\right]. \quad (40)$$

In the case of strong persistence (28), the asymptotic expression for the distribution function Eq. (37) can be written as

$$W_L(k) = W_L(0) \frac{nL}{k(L-k)}, \quad k \neq 0, k \neq L, \quad (41)$$

$$W_L(0) = W_L(L) = \frac{1}{2} (1 - n \ln L). \quad (42)$$

Both the distribution $W_L(k)$ (41) and the function $W_N(k)$ (31) has a concave form. The former assumes the maximal value (42) at the edges of the interval $[0, L]$ and has a minimum at $k=L/2$.

2. Variance $D(L)$

Using the definition Eq. (8) and the distribution function Eq. (37) one can obtain a very simple formula for the variance $D(L)$,

$$D(L) = \frac{L}{4} [1 + m(L-1)], \quad (43)$$

with

$$m = \frac{1}{1+2n} = \frac{2\mu}{N-2\mu(N-1)}. \quad (44)$$

Equation (43) shows that the variance $D(L)$ obeys the parabolic law independently of the correlation strength in the Markov chain.

In the case of weak persistence, at $n \gg 1$, we obtain the asymptotics Eq. (40). It allows one to analyze the behavior

of the variance $D(L)$ with an increase in the “diffusion time” L . At small $mL \ll 1$, the dependence $D(L)$ follows the classical law of the Brownian diffusion, $D(L) \approx L/4$. Then, at $mL \sim 1$, the function $D(L)$ becomes superlinear.

For the case of strong persistence, $n \ll 1$, Eq. (43) gives the asymptotics,

$$D(L) = \frac{L^2}{4} - \frac{nL(L-1)}{2}. \quad (45)$$

The ballistic regime of diffusion leads to the quadratic law of the $D(L)$ dependence in the zero approximation in the parameter $n \ll 1$.

The unusual behavior of the variance $D(L)$ raises an issue as to what particular type of the diffusion equation corresponds to the nonlinear dependence $D(L)$ in Eq. (40). In the following subsection, when solving this problem, we will obtain the conditional probability $p^{(0)}$ of the symbol zero occurring after a given L word with $L < N$. The ability to find $p^{(0)}$, with some reduced information about the preceding symbols being available, is very important for the study of the self-similarity of the Markov chain (see Sec. III B 4).

3. Generalized diffusion equation at $L < N$, $n \gg 1$

It is quite obvious that the distribution $W_L(k)$ satisfies the equation

$$W_{L+1}(k) = W_L(k)p^{(0)}(k) + W_L(k-1)p^{(1)}(k-1). \quad (46)$$

Here $p^{(0)}(k)$ is the probability of “0” occurring after an average-statistical L word containing k unities and $p^{(1)}(k-1)$ is the probability of “1” occurring after an L word containing $(k-1)$ unities. At $L < N$, the probability $p^{(0)}(k)$ can be written as

$$p^{(0)}(k) = \frac{1}{W_L(k)} \sum_{i=k}^{k+N-L} p_i b(i) C_L^k C_{N-L}^{i-k}. \quad (47)$$

The product $b(i) C_L^k C_{N-L}^{i-k}$ in this formula represents the conditional probability of the N word occurring containing i unities, the right-hand part of which, the L subword, contains k unities [compare with Eqs. (35) and (36)].

The product $b(i) C_{N-L}^{i-k}$ in Eq. (47) is a sharp function of i with a maximum at some point $i = i_0$ whereas p_i obeys the linear law (5). This implies that p_i can be factored out of the summation sign being taken at point $i = i_0$. The asymptotical calculation shows that point i_0 is described by the equation,

$$i_0 = \frac{N}{2} - \frac{L/2}{1 - 2\mu(1-L/N)} \left(1 - \frac{2k}{L} \right). \quad (48)$$

Expression (5) taken at point i_0 gives the desired formula for $p^{(0)}$ because

$$\sum_{i=k}^{k+N-L} b(i) C_L^k C_{N-L}^{i-k} \quad (49)$$

is obviously equal to $W_L(k)$. Thus, we have

$$p^{(0)}(k) = \frac{1}{2} + \frac{\mu L}{N - 2\mu(N-L)} \left(1 - \frac{2k}{L} \right). \quad (50)$$

Let us consider a very important point relating to Eq. (48). If the concentration of unities in the right-hand part of the word (36) is higher than $1/2$, $k/L > 1/2$, then the most probable concentration $(i_0 - k)/(N - L)$ of unities in the left-hand part of this word is likewise increased, $(i_0 - k)/(N - L) > 1/2$. At the same time, the concentration $(i_0 - k)/(N - L)$ is less than k/L ,

$$\frac{1}{2} < \frac{i_0 - k}{N - L} < \frac{k}{L}. \quad (51)$$

This implies that the increased concentration of unities in the L words is necessarily accompanied by the existence of a certain tail with an increased concentration of unities as well. Such a phenomenon is referred to by us as the *macro-persistence*. An analysis performed in the following section will indicate that the correlation length l_c of this tail is γN with $\gamma \geq 1$ dependent on the parameter μ only. It is evident from the above-mentioned property of the isotropy of the Markov chain that there are two correlation tails from both sides of the L word.

Note that the distribution $W_L(k)$ is a smooth function of arguments k and L near its maximum in the case of weak persistence and $k, L - k \gg 1$. By going over to the continuous limit in Eq. (46) and using Eq. (50) with the relation $p^{(1)}(k-1) = 1 - p^{(0)}(k-1)$, we obtain the diffusion Fokker-Planck equation for the correlated Markov process,

$$\frac{\partial W}{\partial L} = \frac{1}{8} \frac{\partial^2 W}{\partial \kappa^2} - \eta(L) \frac{\partial}{\partial \kappa} (\kappa W), \quad (52)$$

where $\kappa = k - L/2$ and

$$\eta(L) = \frac{2\mu}{(1 - 2\mu)N + 2\mu L}. \quad (53)$$

Equation (52) has a solution of the Gaussian form Eq. (39) with the variance $D(L)$ satisfying the ordinary differential equation,

$$\frac{dD}{dL} = \frac{1}{4} + 2\eta(L)D. \quad (54)$$

Its solution, given the boundary condition $D(0) = 0$, coincides with Eq. (40).

4. Self-similarity of the persistent Brownian diffusion

In this subsection, we point to one of the most interesting properties of the Markov chain being considered, namely, its self-similarity. Let us reduce the N -step Markov sequence by regularly (or randomly) removing some symbols and introduce the decimation parameter λ ,

$$\lambda = N^*/N \leq 1. \quad (55)$$

Here N^* is a renormalized memory length for the reduced N^* -step Markov chain. According to Eq. (50), the conditional probability p_k^* of occurring the symbol zero after k unities among the preceding N^* symbols is described by the formula

$$p_k^* = \frac{1}{2} + \mu^* \left(1 - \frac{2k}{N^*} \right), \quad (56)$$

with

$$\mu^* = \mu \frac{\lambda}{1 - 2\mu(1 - \lambda)}. \quad (57)$$

The comparison between Eqs. (5) and (56) shows that the reduced chain possesses the same statistical properties as the initial one but it is characterized by the renormalized parameters (N^*, μ^*) instead of (N, μ) . Thus, Eqs. (55) and (57) determine the one-parametrical renormalization of the parameters of the stochastic process defined by Eq. (5).

The astonishing property of the reduced sequence consists in that *the variance $D^*(L)$ is invariant with respect to the one-parametric decimation transformation* (55) and (57). In other words, it coincides with the function $D(L)$ for the initial Markov chain:

$$D^*(L) = \frac{L}{4} [1 + m^*(L-1)] = D(L), \quad L < N^*. \quad (58)$$

Indeed, according to Eqs. (55) and (57), the renormalized parameter $m^* = 2\mu^*/[N^* - 2\mu^*(N^* - 1)]$ of the reduced sequence coincides exactly with the parameter $m = 2\mu/[N - 2\mu(N - 1)]$ of the initial Markov chain. Since the shape of the function $W_L(k)$ Eq. (37) is defined by the invariant parameter $n = n^*$, the distribution $W_L(k)$ is also invariant with respect to the decimation transformation.

The transformation $(N, \mu) \rightarrow (N^*, \mu^*)$ Eqs. (55) and (57), possesses the properties of semigroup, i.e., the composition of transformations $(N, \mu) \rightarrow (N^*, \mu^*)$ and $(N^*, \mu^*) \rightarrow (N^{**}, \mu^{**})$ with transformation parameters λ_1 and λ_2 is likewise the transformation from the same semigroup, $(N, \mu) \rightarrow (N^{**}, \mu^{**})$, with parameter $\lambda = \lambda_1 \lambda_2$.

The invariance of the function $D(L)$ at $L < N$ was referred to by us as the phenomenon of *self-similarity*. It is demonstrated in Fig. 3 and is accordingly discussed below, in Sec. IV A.

It is interesting to note that the property of self-similarity is valid for any strength of the persistence. Indeed, the result Eq. (50) can be obtained directly from Eqs. (15)–(17), and Eq. (47) not only for $n \gg 1$ but also for the arbitrary value of n .

C. Long-range diffusion, $L > N$

Unfortunately, the very useful proposition 1 is valid for the words of the length $L \leq N$ only and is not applicable to the analysis of the long words with $L > N$. Therefore, investigating the statistical properties of the long words represents a rather challenging combinatorial problem and requires new

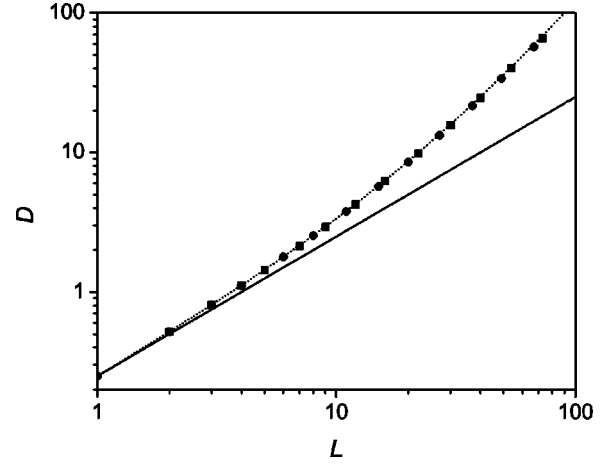


FIG. 3. The dependence of the variance D on the tuple length L for the generated sequence with $N=100$ and $\mu=0.4$ (dotted line) and for the decimated sequences (the parameter of decimation $\lambda=0.5$). Squares and circles correspond to the stochastic and deterministic reduction, respectively. The solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

physical approaches for its simplification. Thus, we start this subsection by analyzing the correlation properties of the long words ($L > N$) in the Markov chains with $N \gg 1$. The two first subsections of this subsection mainly deal with the case of relatively weak correlations, $n \gg 1$.

1. Correlation length at weak persistence

Let us rewrite Eq. (5) in the form

$$\langle a_{i+1} \rangle = \frac{1}{2} + \mu \left(\frac{2}{N} \sum_{k=i-N+1}^i \langle a_k \rangle - 1 \right). \quad (59)$$

The angle brackets denote the averaging of the density of unities in some region of the Markov chain for its definite realization. The averaging is performed over distances much greater than unity but far less than the memory length N and correlation length l_c [see Eq. (63) below]. Note that this averaging differs from the statistical averaging over the ensemble of realizations of the Markov chain denoted by the bar in Eqs. (8) and (9). Equation (59) is a relationship between the average densities of unities in two different macroscopic regions of the Markov chain, namely, in the vicinity of $(i+1)$ th element and in the region $(i-N, i)$. Such an approach is similar to the mean field approximation in the theory of the phase transitions and is asymptotically exact at $N \rightarrow \infty$. In the continuous limit, Eq. (59) can be rewritten in the integral form,

$$\langle a(i) \rangle = \frac{1}{2} + \mu \left(\frac{2}{N} \int_{i-N}^i \langle a(k) \rangle dk - 1 \right). \quad (60)$$

It has the obvious solution,

$$\left\langle a(i) - \frac{1}{2} \right\rangle = \left\langle a(0) - \frac{1}{2} \right\rangle \exp(-i/\gamma N), \quad (61)$$

where the parameter γ is determined by the relation,

$$\gamma \left(\exp\left(\frac{1}{\gamma}\right) - 1 \right) = \frac{1}{2\mu}. \quad (62)$$

A unique solution γ of the last equation is an increasing function of $\mu \in (0, 1/2)$.

Formula (61) shows that any fluctuation [the difference between $\langle a(i) \rangle$ and the equilibrium value of $a_i = 1/2$] is exponentially damped at distances of the order of the *correlation length* l_c ,

$$l_c = \gamma N. \quad (63)$$

Law (61) describes the phenomenon of the *persistent macroscopic correlations* discussed in the previous subsection. This phenomenon is governed by both parameters, N and μ . According to Eqs. (62) and (63), the correlation length l_c grows as $\gamma = 1/4\delta$ with an increase in μ (at $\mu \rightarrow 1/2$) until the inequality $\delta \gg 1/N$ is satisfied. Here

$$\delta = 1/2 - \mu. \quad (64)$$

Let us note that the inequality $\delta \gg 1/N$ defining the regime of weak persistence can be rewritten in terms of γ , $\gamma \ll N/4$. At $\delta \approx 1/N$, the correlation length l_c achieves its maximum value $N^2/4$. With the following increase of μ , the diffusion goes to the regime of strongly correlated diffusion that will be discussed in Sec. 3 III C.

At $\mu \rightarrow 0$, the macropersistence is broken and the correlation length tends to zero.

2. Correlation function at weak persistence

Using the studied correlation properties of the Markov sequence and some heuristic reasons, one can obtain the correlation function $\mathcal{K}(r)$ being defined as

$$\mathcal{K}(r) = \overline{a_i a_{i+r}} - \overline{a_i}^2, \quad (65)$$

and then the variance $D(L)$. Comparing Eq. (65) with Eqs. (7) and (8) and taking into account the property of sequence, $\overline{a_i} = 1/2$, it is easy to derive the general relationship between functions $\mathcal{K}(r)$ and $D(L)$,

$$D(L) = \frac{L}{2} + 4 \sum_{i=1}^{L-1} \sum_{r=1}^{L-i} \mathcal{K}(r). \quad (66)$$

Considering Eq. (66) as an equation with respect to $\mathcal{K}(r)$, one can find its solution,

$$\begin{aligned} \mathcal{K}(1) &= \frac{1}{2}D(2) - \frac{1}{4}, \quad \mathcal{K}(2) = \frac{1}{2}D(3) - D(2) + \frac{1}{8}, \\ \mathcal{K}(r) &= \frac{1}{2}[D(r+1) - 2D(r) + D(r-1)], \quad r \geq 3. \end{aligned} \quad (67)$$

This solution has a very simple form in the continuous limit, with

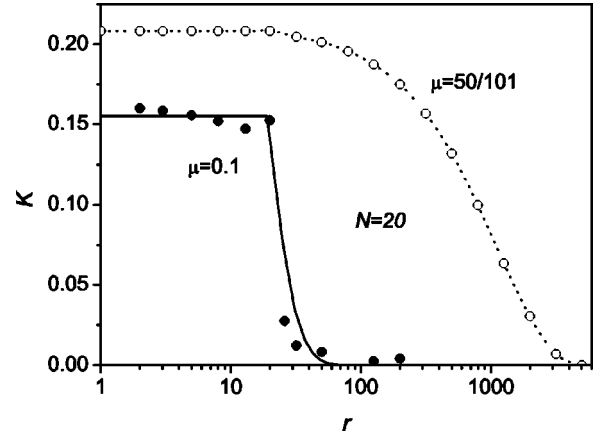


FIG. 4. The dependence of the correlation function K on the distance r between the symbols for the sequence with $N=20$. The dots correspond to the generated sequence with $\mu=0.1$ and $\mu=50/101$. The solid line is analytical result (70) with $l_c = \gamma N$ and $\gamma = 0.38$.

$$\mathcal{K}(r) = \frac{1}{2} \frac{d^2 D(r)}{dr^2}. \quad (68)$$

Equations (67) and (43) give the correlation function at $r < N$, $n \geq 1$,

$$\mathcal{K}(r) = C_r m,$$

with

$$C_1 = 1/2, \quad C_2 = 1/8, \quad C_{3 \leq r \leq N} = 1/4,$$

and m determined by Eq. (44). In the continuous approximation, the correlation function is described by the formula

$$\mathcal{K}(r) = \frac{m}{4}, \quad r \leq N. \quad (69)$$

The independence of the correlation function of r at $r < N$ results from our choice of the conditional probability in the simplest form (5). At $r > N$, the function $\mathcal{K}(r)$ should decrease because of the loss of memory. Therefore, using Eqs. (61) and (63), let us prolongate the correlator $\mathcal{K}(r)$ as the exponentially decreasing function at $r > N$,

$$\mathcal{K}(r) = \frac{m}{4} \begin{cases} 1, & r \leq N, \\ \exp\left(-\frac{r-N}{l_c}\right), & r > N. \end{cases} \quad (70)$$

The solid line in Fig. 4 presents the plot of the correlation function at $\mu = 0.1$. The value of the correlation function for this curve is increased by multiplier 50.

According to Eqs. (68) and (70), the variance $D(L)$ can be written as

$$D(L) = \frac{L}{4} (1 + mF(L)), \quad (71)$$

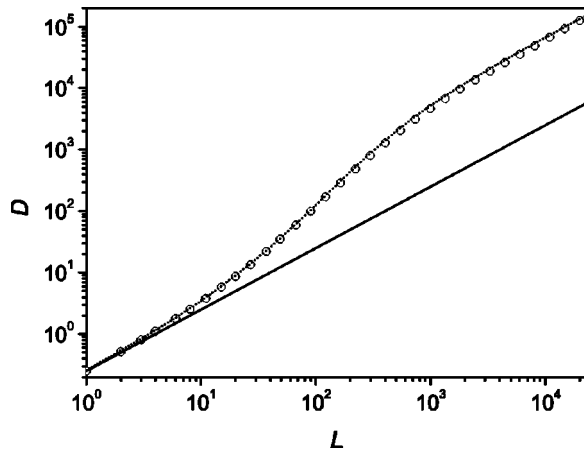


FIG. 5. The numerical simulation of the dependence $D(L)$ for the generated sequence with $N=100$ and $\mu=0.4$ (circles). The dotted line is the plot of function Eq. (71) with the same values of N and μ . The solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

$$F(L) = \begin{cases} L, & L < N, \\ 2(1+\gamma)N - (1+2\gamma)\frac{N^2}{L}, & \\ -2\gamma^2\frac{N^2}{L}\left[1 - \exp\left(-\frac{L-N}{l_c}\right)\right], & L > N. \end{cases} \quad (72)$$

As an illustration of the result Eq. (71), we present the plot of $D(L)$ for $N=100$ and $\mu=0.4$ by the dotted line in Fig. 5. The straight line in the figure corresponds to the dependence $D(L)=L/4$ for the usual Brownian diffusion without correlations (for $\mu=0$). It is clearly seen that the plot of variance (71) contains two qualitatively different portions. One of them, at $L \leq N$, is the superlinear curve that moves away from the line $D=L/4$ with an increase of L as a result of the persistence. For $L \geq N$, the curve $D(L)$ achieves the linear asymptotics,

$$D(L) \cong \frac{L}{4} \left(1 + \frac{4\mu(1+\gamma)}{1-2\mu} \right). \quad (73)$$

This phenomenon can be interpreted as a result of the diffusion in which every *independent* step $\sim \sqrt{D(L)}$ of wandering represents a path traversed by a particle during the characteristic “fluctuating time” $L \sim (N+l_c)$. Since these steps of wandering are quasi-independent, the distribution function $W_L(k)$ is the Gaussian. Thus, in the case of relatively weak persistence, $n \geq 1$, $W_L(k)$ is the Gaussian not only at $L < N$ [see Eq. (39)] but also for $L > N$, l_c .

Note that the above-mentioned property of the self-similarity is valid only at the portion $L < N$ of the curve $D(L)$. Since the decimation procedure leads to the decrease of the parameter μ [see Eq. (57)], the plot of asymptotics (73) for the reduced sequence at $L \geq N^*$ goes below the $D(L)$ plot for the initial chain.

3. Statistics of the L words for the case of strong persistence, $n \ll \ln^{-1}N$

In this subsection, we study the statistical properties of long words ($L > N$) in the sequences of symbols with strong correlations. It is convenient to rewrite formula (5) for the conditional probability of occurring the symbol zero after the N word containing k unities in the form,

$$p_\nu = \delta + 2\mu \frac{\nu}{N}, \quad (74)$$

where ν is the number of zeros in the precedent N word, $\nu = N - k$.

In the case of strong persistence, $n \ll \ln^{-1}N$, the parameter $\delta = 1/2 - \mu$ is much smaller than $1/N$. Therefore, the probability p_ν can be written as

$$p_\nu \approx \begin{cases} \delta, & \nu = 0, \\ \nu/N, & \nu \neq 0, \nu \neq N, \\ 1 - \delta, & \nu = N. \end{cases} \quad (75)$$

It is seen that the probability of occurring the symbol zero after the N word which contains only unities ($\nu=0$) represents very small value δ and it increases significantly if $\nu \neq 0$. This situation differs drastically from the case of weak persistency. At $n \geq 1$, the parameter δ exceeds noticeably the value $1/N$, and the probability p_ν does not actually change with an increase in the number of zeros in the preceding N word.

The analysis of the symbol generation process in the Markov chain in the case of strong persistence gives the following picture of the fluctuations. There exist the entire portions of the chain consisting of the same symbols, say unities. The characteristic length of such portions is $1/\delta \gg N$. These portions are separated by one or more symbols zero. The number of such packets of the same symbols in one fluctuation zone is about N . Thus, the characteristic correlation distance at which the N word containing the same symbols converts into the N word with $\nu = N/2$ is about N/δ ,

$$l_c \approx \frac{N}{\delta}. \quad (76)$$

The described structure of the fluctuations defines the statistical properties of the L words with $L > N$ in the case of strong persistence. The distribution function differs significantly from the Gaussian and is characterized by a concave form at $L \leq l_c \sim N/\delta$. As L increases, the correlations between different parts of the L words get weaker and the L word can be considered as consisting of a number of independent subwords. So, according to the general mathematical theorems [12,17], the distribution function takes on the usual Gaussian form. Such an evolution of the distribution function is depicted in Fig. 6. Note that the Gaussian form of the distribution function is the direct consequence of the finiteness of the correlation length. Thus the Markov chains studied in this paper differ essentially from strongly correlated systems with $l_c \rightarrow \infty$, which are widely discussed in litera-

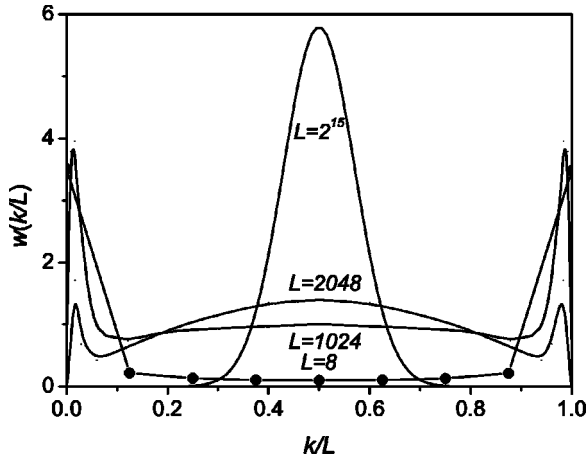


FIG. 6. The distribution function $w(k/L)=LW_L(k)$ for $N=8$ and $\delta=1/150$. Different values of the length L of words is shown near the curves.

ture. As is known [11], such systems are characterized by the distribution functions with the power-law tails (the so-called Levy distributions).

The variance $D(L)$ follows the quadratic law $D=L^2/4$ [see Eq. (45)] up to the range of $L \leq l_c \sim N/\delta$ and then approaches to the asymptotics $D(L)=BL$ with $B \sim N/4\delta$ (see Fig. 7).

The dotted line in Fig. 4 presents the correlation function for the case of strong persistence ($\mu = 50/101$, $N=20$).

IV. RESULTS OF NUMERICAL SIMULATIONS AND APPLICATIONS

In this section, we support the obtained analytical results by numerical simulations of the Markov chain with the conditional probability Eq. (5). Besides, the properties of the studied binary N -step Markov chain are compared with those for the natural objects, specifically for the coarse-grained written and DNA texts.

A. Numerical simulations of the Markov chain

The first stage of the construction of the N -step Markov chain was a generation of the initial noncorrelated N sym-

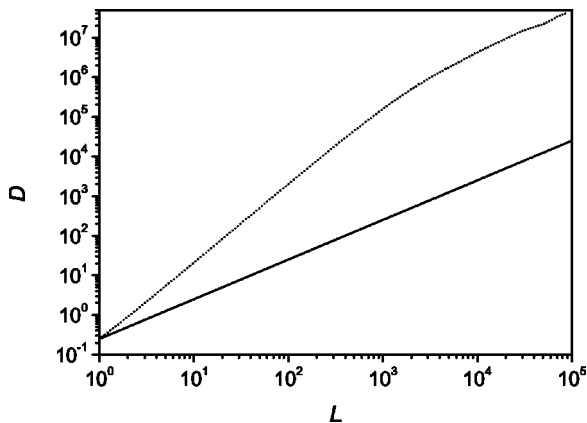


FIG. 7. The dependence of the variance D on the word length L for the sequence with $N=20$ and $\mu = 50/101$ (dotted line). The solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

bols, zeros and unities, identically distributed with equal probabilities 1/2. Each consequent symbol was then added to the chain with the conditional probability determined by the previous N symbols in accordance with Eq. (5). Then we numerically calculated the variance $D(L)$ by means of Eq. (8). The circles in Fig. 5 represent the calculated variance $D(L)$ for the case of weak persistence ($n = 12.5 \gg 1$). A very good agreement between the analytical result (71) and the numerical simulation can be observed. The case of strong persistence is illustrated by Figs. 6 and 7 where the distribution function $W_L(k)$ and the variance $D(L)$ are calculated numerically for $n=4/37$ and $n=0.1$, respectively. The circles on the curves in Fig. 4 represent the calculated results for the correlation function $\mathcal{K}(r)$ for $n=0.1$ (on the dotted line) and $n=40$ (near the solid line).

The numerical simulation was also used for the demonstration of the proposition 1 (Fig. 1) and the self-similarity property of the Markov sequence (Fig. 3). The squares in Fig. 3 represent the variance $D(L)$ for the sequence obtained by the stochastic decimation of the initial Markov chain (dotted line) where each symbol was omitted with the probability 1/2. The circles in this figure correspond to the regular reduction of the sequence by removing each second symbol.

And finally, the numerical simulations have allowed us to make sure that we are able to determine the parameters N and μ of a given binary sequence. We generated the Markov sequences with different parameters N and μ and defined numerically the corresponding curves $D(L)$. Then we solved the inverse problem of the reconstruction of the parameters N and μ by analyzing the curves $D(L)$. The reconstructed parameters were always in good agreement with their prescribed values. In the following subsections we apply this ability to the treatment of the statistical properties of literary and DNA texts.

B. Literary texts

It is well known that the statistical properties of the coarse-grained texts written in any language exhibit a remarkable deviation from random sequences [4,18]. In order to check the applicability of the theory of the binary N -step Markov chains to literary texts, we resorted to the procedure of coarse graining by the random mapping of all characters of the text onto the binary set of symbols, zeros and unities. The statistical properties of the coarse-grained texts depend, but not significantly, on the kind of mapping. This is illustrated by the curves in Fig. 8 where the variance $D(L)$ for five different kinds of the mapping of Bible is presented. In general, the random mapping leads to nonequal numbers of unities and zeros, k_1 and k_0 , in the coarse-grained sequence. A particular analysis indicates that the variance $D(L)$ (40) gets the additional multiplier,

$$\frac{4k_0k_1}{(k_0+k_1)^2},$$

in this biased case. In order to derive the function $D(L)$ for the nonbiased sequence, we divided the numerically calculated value of the variance by this multiplier.

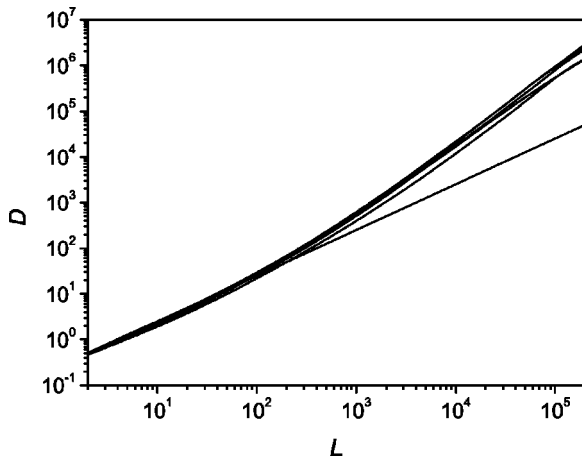


FIG. 8. The dependence $D(L)$ for the coarse-grained text of Bible obtained by means of five different kinds of random mapping. The straight solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

The study of different written texts has suggested that all of them feature the pronounced persistent correlations. It is demonstrated by Fig. 9 where the five variance curves go significantly higher than the straight line $D=L/4$. However, it should be emphasized that regardless of the kind of mapping the initial portions, $L < 80$, of the curves correspond to a slight antipersistent behavior (see inset to Fig. 10). Moreover, for some inappropriate kinds of mapping (e.g., when all vowels are mapped onto the same symbol) the antipersistent portions can reach the values of $L \sim 1000$. To avoid this problem, all the curves in Fig. 9 are obtained for the definite representative mapping: $(a-m) \rightarrow 0$; $(n-z) \rightarrow 1$.

Thus the persistence is the common property of the binary N -step Markov chains that have been considered in this pa-

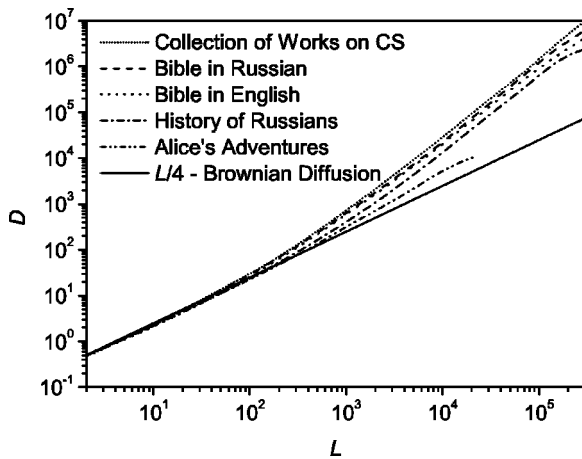


FIG. 9. The dependence $D(L)$ for the coarse-grained texts of collection of works on the computer science ($m=2.2 \times 10^{-3}$, short-dotted line), Bible in Russian ($m=1.9 \times 10^{-3}$, dashed line), Bible in English ($m=1.5 \times 10^{-3}$, dotted line), "History of Russians in the 20th Century" by Oleg Platonov ($m=6.4 \times 10^{-4}$, dash-dotted line), and "Alice's Adventures in Wonderland" by Lewis Carroll ($m=2.7 \times 10^{-4}$, dash-dot-dotted line). The solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

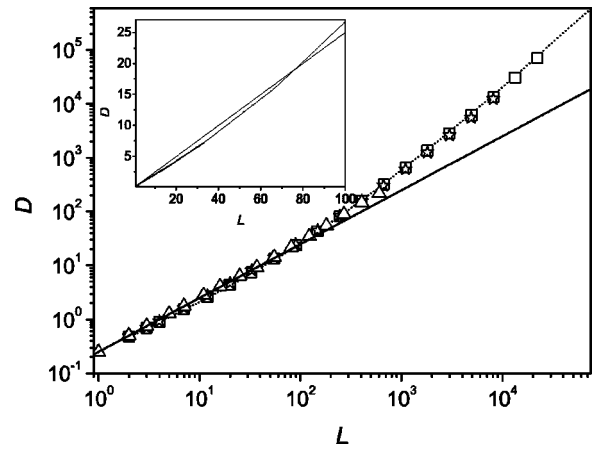


FIG. 10. The dependence of the variance D on the tuple length L for the coarse-grained text of Bible (dotted line) and for the decimated sequences with different parameters λ : $\lambda=3/4$ (squares), $\lambda=1/2$ (stars), and $\lambda=1/256$ (triangles). The solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$. The inset demonstrates the antipersistent portion of the $D(L)$ plot for Bible.

per and the coarse-grained written texts at large scales. Moreover, the written texts as well as the Markov sequences possess the property of the self-similarity. Indeed, the curves in Fig. 10 obtained from the text of Bible with different levels of the deterministic decimation demonstrate the self-similarity. Presumably, this property is the mathematical reflection of the well-known hierarchy in the linguistics: *letters* \rightarrow *syllables* \rightarrow *words* \rightarrow *sentences* \rightarrow *paragraphs* \rightarrow *chapters* \rightarrow *books*.

All the above-mentioned circumstances allow us to suppose that our theory of the binary N -step Markov chains can be applied to the description of the statistical properties of the texts of natural languages. However, in contrast to the generated Markov sequence (see Fig. 5) where the full length \mathcal{M} of the chain is far greater than the memory length N , the coarse-grained texts described by Fig. 9 are of relatively short length $\mathcal{M} \lesssim N$. In other words, the coarse-grained texts are similar not to the Markov chains but rather to some non-stationary short fragments. This implies that each of the written texts is correlated throughout the whole of its length. Therefore, as far as the written texts are concerned, it is impossible to observe the second portion of the curve $D(L)$ parallel (in the log-log scale) to the line $D(L)=L/4$, similar to that shown in Fig. 5. As a result, one cannot define the values of both parameters N and μ for the coarse-grained texts. The analysis of the curves in Fig. 6 can give the combination $m=2\mu/N(1-2\mu)$ only [see Eq. (40)]. Perhaps, this particular combination is the real parameter governing the persistent properties of the literary texts.

We would like to note that the origin of the long-range correlations in the literary texts is hardly related to the grammatical rules as is claimed in Ref. [4]. At short scales $L \leq 80$ where the grammatical rules are in fact applicable the character of correlations is antipersistent (see the inset in Fig. 10) whereas semantic correlations lead to the global persistent behavior of the variance $D(L)$ throughout the entire of literary text.

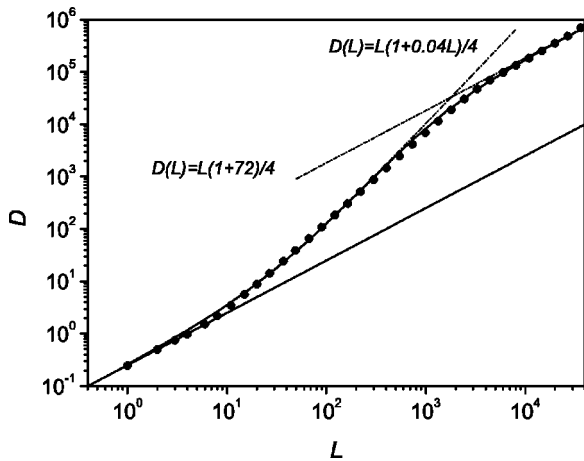


FIG. 11. The dependence $D(L)$ for the coarse-grained alphabetical list of 15 462 English words (circles). The curved solid line is the plot of function Eq. (55) with the fitting parameters $N=180$ and $\mu=0.44$. The straight solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

The numerical estimations of the persistent parameter m and the characterization of the languages and different authors using this parameter can be regarded as a new intriguing problem of linguistics. For instance, the unprecedented low value of m for the very inventive work by Lewis Carroll as well as the closeness of m for the texts of English and Russian versions of Bible are of certain interest.

It should be noted that there exist special kinds of short-range correlated texts which can be specified by both of the parameters, N and μ . For example, all dictionaries consist of the families of words where some preferable letters are repeated more frequently than in their other parts. Yet another example of the shortly correlated texts is any lexicographically ordered list of words. The analysis of written texts of this kind is given below.

C. Dictionaries

As an example, we have investigated the statistical properties of the coarse-grained alphabetical (lexicographically ordered) list of the most frequently used 15462 English words. In contrast to other texts, the statistical properties of the coarse-grained dictionaries are very sensitive to the kind of mapping. If one uses the above-mentioned mapping, $(a-m) \rightarrow 0; (n-z) \rightarrow 1$, the behavior of the variance $D(L)$ similar to that shown in Fig. 9 would be obtained. The particular construction of the dictionary manifests itself if the preferable letters in the neighboring families of words are mapped onto the different symbols. The variance $D(L)$ for the dictionary coarse-grained by means of such mapping is shown by circles in Fig. 11. It is clearly seen that the graph of the function $D(L)$ consists of two portions similar to the curve in Fig. 5 obtained for the generated N -step Markov sequence. The fitting of the curve in Fig. 11 by function (71) (the curved solid line in Fig. 11) yielded the values of the parameters $N=180$ and $\mu=0.44$.

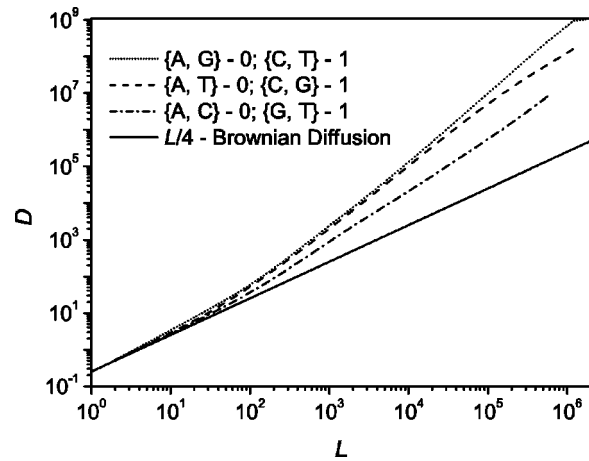


FIG. 12. The dependence $D(L)$ for the coarse-grained DNA text of *Bacillus subtilis*, complete genome, for three nonequivalent kinds of mapping. Dotted, dashed, and dash-dotted lines correspond to the mappings $\{A,G\} \rightarrow 0, \{C,T\} \rightarrow 1$ (the parameter $m=4.1 \times 10^{-2}$), $\{A,T\} \rightarrow 0, \{C,G\} \rightarrow 1$ ($m=2.5 \times 10^{-2}$), and $\{A,C\} \rightarrow 0, \{G,T\} \rightarrow 1$ ($m=1.5 \times 10^{-2}$), respectively. The solid line describes the noncorrelated Brownian diffusion, $D(L)=L/4$.

D. DNA texts

It is known that any DNA text is written by four “characters,” specifically by adenine (A), cytosine (C), guanine (G), and thymine (T). Therefore, there are three nonequivalent types of the DNA text mapping onto one-dimensional binary sequences of zeros and unities. The first of them is the so-called purine-pyrimidine rule, $\{A,G\} \rightarrow 0, \{C,T\} \rightarrow 1$. The second one is the hydrogen-bond rule, $\{A,T\} \rightarrow 0, \{C,G\} \rightarrow 1$. And, finally, the third is $\{A,C\} \rightarrow 0, \{G,T\} \rightarrow 1$.

By way of example, the variance $D(L)$ for the coarse-grained text of *Bacillus subtilis*, complete genome (ftp://ftp.ncbi.nih.gov/genomes/bacteria/bacillus_subtilis/NC_000964.gbk) is displayed in Fig. 12 for all possible types of mapping. One can see that the persistent properties of DNA are more pronounced than for the written texts and, contrary to the written texts, the $D(L)$ dependence for DNA does not exhibit the antipersistent behavior at small values of L . In our view, the noticeable deviation of different curves in Fig. 12 from each other demonstrates that the DNA texts are much more complex objects in comparison with the written ones. Indeed, the different kinds of mapping reveal and emphasize various types of physical attractive correlations between the nucleotides in DNA, such as the strong purine-purine and pyrimidine-pyrimidine persistent correlations (the upper curve), and the correlations caused by a weaker attraction $A \leftrightarrow T$ and $C \leftrightarrow G$ (the middle curve).

It is interesting to compare the correlation properties of the DNA texts for three different species that belong to the major domains of living organisms: the Bacteria, the Archaea, and the Eukarya [19]. Figure 13 shows the variance $D(L)$ for the coarse-grained DNA texts of *Bacillus subtilis* (the Bacteria), *Methanosarcina acetivorans* (the Archaea), and *Drosophila melanogaster*—fruit fly—(the Eukarya) for the most representative mapping $\{A,G\} \rightarrow 0, \{C,T\} \rightarrow 1$. It is seen that the $D(L)$ curve for the DNA text of *Bacillus sub-*

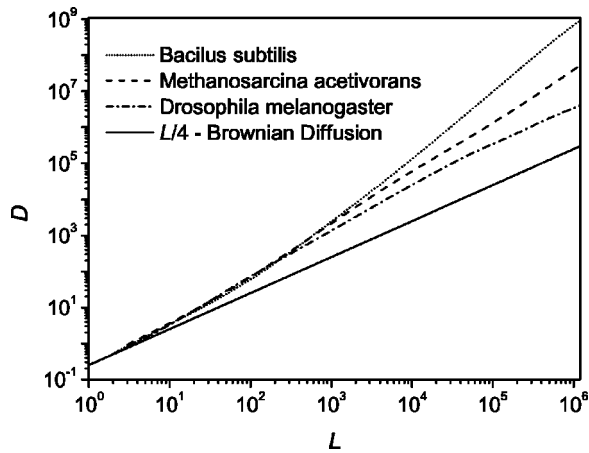


FIG. 13. The dependence $D(L)$ for the coarse-grained DNA texts of *Bacillus subtilis*, complete genome, the Bacteria (dotted line); *Methanosarcina acetivorans*, complete genome, the Archaea (dashed line); *Drosophila melanogaster* chromosome X, complete sequence, the Eukarya (dash-dotted line) for the mapping $\{A,G\} \rightarrow 0$, $\{C,T\} \rightarrow 1$. The solid line describes the noncorrelated Brownian diffusion, $D(L) = L/4$.

tilis is characterized by the highest persistence. As well as for the written texts, the $D(L)$ curves for the DNA of both the Bacteria and the Archaea do not contain the linear portions given by Eq. (73). This suggests that their DNA chains are not stationary sequences. In this connection, we would like to point out that their DNA molecules are circular and represent the collection of extended coding regions interrupted by small noncoding regions. According to Figs. 12 and 13, the noncoding regions do not disrupt the correlation between the coding areas, and the DNA systems of the Bacteria and the Archaea are fully correlated throughout their entire lengths. Contrary to them, the DNA molecules of the Eukarya have the linear structure and contain long noncoding portions. As evident from Fig. 13, the DNA sequence of the representative of the Eukarya is not entirely correlated. The $D(L)$ curve for the X chromosome of the fruit fly corresponds qualitatively to Eqs. (71) and (72) with $\mu \approx 0.35$ and $N \approx 250$. If one draws an analogy between the DNA sequences and the literary texts, the resemblance of the correlation properties of integral literary novels and the DNA texts of the Bacteria and Archaea are to be found, whereas the

DNA texts of the Eukarya are more similar to the collections of 10^4 – 10^5 short stories.

V. CONCLUSION

Thus, we have developed an approach to describing the strongly correlated one-dimensional systems. The simple, exactly solvable model of the uniform binary N -step Markov chain is presented. The memory length N and the parameter μ of the persistent correlations are two parameters in our theory. The correlation function $\mathcal{K}(r)$ is usually employed as the input characteristics for the description of the correlated random systems. Yet, the function $\mathcal{K}(r)$ describes not only the direct interconnection of the elements a_i and a_{i+r} , but also takes into account their indirect interaction via other elements. Since our approach operates with the “original” parameters N and μ , we believe that it allows us to reveal the intrinsic properties of the system which provide the correlations between the elements.

We have demonstrated the applicability of the developed theoretical model to the different kinds of relatively weakly correlated stochastic systems. Perhaps, the case of strong persistency is also of interest from the standpoint of possible applications. Indeed, the domain structure of the symbol fluctuations at $n \ll 1$ is very similar to the domains in magnetics. Thus, an attempt to model the magnetic structures by the Markov chains with strongly pronounced persistent properties can be appropriate.

We would like to note that there exist some features of the real correlated systems which cannot be interpreted in terms of our two-parametric model. For example, the interference of the grammatical antipersistent and semantic persistent correlations in the literary texts requires more than two parameters for their description. Obviously, more complex models should be worked out for the adequate interpretation of the statistical properties of the DNA texts and other real correlated systems. In particular, the Markov chains consisting of more than two different elements (non-binary chains) can be suitable for modeling the DNA systems.

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